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On Nonlinear Discrete Boundary Value Problems

JESUS RODRIGUEZ

Department of Mathematics, Box 8205, North Carolina State University, Raleigh, North Carolina 27695-8205

Submitted by E. Stanley Lee

1. INTRODUCTION

As both discretized versions of differential equations and as models in their own right boundary value problems for discrete dynamical systems have been extensively studied the last several years [2, 3, 5, 10, 17, 19].

In this paper we consider discrete multipoint boundary value problems of the forms:

$$x(t+1) = A(t)\,x(t) + h(t) + \lambda f(t, x(t)), \qquad t \in \{0, 1,...\} \tag{1.1}$$

subject to

$$\sum_{j=0}^{N} B_j x(j) = v + \lambda g(x(0),..., x(N)) \tag{1.2}$$

and

$$x(t+1) = f(t, x(t)), \qquad t \in \{0, 1,...\} \tag{1.3}$$

subject to

$$g(x(0), x(1),..., x(N)) = 0 \tag{1.4}$$

where N is a fixed positive integer; $B_0,..., B_N$ represent n by n constant matrices; $A(t)$ is an n by n matrix for each $t \in \{0, 1,...\}$; $h(t)$ belongs to $\mathbb{R}^n$ for each nonnegative integer t; v is in $\mathbb{R}^n$; λ is a real parameter; f and g are continuously differentiable, f: $\mathbb{R}^{n+1} \to \mathbb{R}^n$ and g: $\mathbb{R}^{n(N+1)} \to \mathbb{R}^n$.

For systems (1.1)–(1.2) we prove that if the linear system

$$x(t+1) = A(t)\,x(t) + h(t) \tag{1.5}$$

subject to

$$\sum_{j=0}^{N} B_j x(j) = v \tag{1.6}$$

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has a unique solution for each pair $h(\cdot), v$ then the same is true for (1.1)–(1.2) provided λ is sufficiently small.

For systems (1.3)–(1.4) sufficient conditions are established for the existence of an isolated solution based on the existence of an "approximate" solution. This extends previous results of Agarwal [2] to the case of nonlinear boundary conditions.

2. Preliminaries

If the n by n matrix $A(t)$ is defined for each nonnegative integer t the state-transition matrix [2, 15] is given by

$$\begin{aligned} \Phi(t, l) &= A(t-1)\, A(t-2) \cdots A(l), & t > l \\ &= I, & t = l. \end{aligned}$$

It is well known [2] that

$$\Phi(t+1, l) = A(t)\, \Phi(t, l) \qquad \text{for } t \geqslant l$$

and that the general solution of

$$x(t+1) = A(t)\, x(t) + h(t)$$

is given by

$$x(t) = \Phi(t, 0)\, x(0) + \sum_{j=0}^{t-1} \Phi(t, j+1)\, h(j) \qquad \text{for } t > 0.$$

The following result appears in [2].

LEMMA 2.1. *If $\Delta = \sum_{j=0}^{N} B_j \Phi(j, 0)$ is nonsingular then for each pair $\binom{h(\cdot)}{v}$ there exists a unique solution of*

$$x(t+1) = A(t)\, x(t) + h(t)$$

subject to

$$\sum_{j=0}^{N} B_j x(j) = v.$$

We now introduce the following notation: $X = \{\phi\colon \{0, 1, ..., N+1\} \to \mathbb{R}^n\}$ and $Z = \{\psi\colon \{0, 1, ..., N\} \to \mathbb{R}^n\}$. For $\phi \in X$ we define $\|\phi\| = \max_{t \in \{0,1,...,N+1\}} |\phi(t)|$ and if $\psi \in Z$ we set $\|\psi\| = \max_{t \in \{0,1,...,N\}} |\psi(t)|$, where $|\cdot|$ denotes any norm on $\mathbb{R}^n$. If $\binom{h(\cdot)}{v}$ is an element of $Z \times \mathbb{R}^n$ we define $\|\binom{h}{v}\| = \|h\| + |v|$.

It is clear that with these norms X, Z, and $Z \times \mathbb{R}^n$ become Banach spaces.

It should be observed that if Δ is nonsingular Lemma 2.1 establishes the existence of a bounded linear map $H: Z \times \mathbb{R}^n \to X$ such that for each $\binom{h}{v}$ in $Z \times \mathbb{R}^n$, $H\binom{h}{v}$ is the unique solution of (1.5)–(1.6). As the following argument illustrates a simple formula can be obtained for the linear map H.

Clearly, $x(\cdot)$ solves problem (1.5)–(1.6) if and only if

$$x(t) = \Phi(t, 0)\, x(0) + \sum_{j=0}^{t-1} \Phi(t, j+1)\, h(j) \qquad \text{for } t > 0$$

and

$$v = \sum_{j=0}^{N} B_j x(j).$$

Equivalently, $x(\cdot)$ solves (1.5)–(1.6) if and only if

$$\begin{aligned} v &= B_0 x(0) + \sum_{j=1}^{N} B_j \left\{ \Phi(j, 0)\, x(0) + \sum_{l=0}^{j-1} \Phi(j, l+1)\, h(l) \right\} \\ &= B_0 x(0) + \sum_{j=1}^{N} B_j \Phi(j, 0)\, x(0) + \sum_{j=1}^{N} B_j \sum_{l=0}^{j-1} \Phi(j, l+1)\, h(l) \\ &= \Delta x(0) + \sum_{j=1}^{N} B_j \sum_{l=0}^{j-1} \Phi(j, l+1)\, h(l). \end{aligned}$$

Since Δ is nonsingular we obtain

$$x(0) = \Delta^{-1} \left\{ v - \sum_{j=1}^{N} B_j \sum_{l=0}^{j-1} \Phi(j, l+1)\, h(l) \right\}$$

and hence, $H: Z \times \mathbb{R}^n \to X$ is given by

$$\begin{aligned} H\begin{pmatrix} h \\ v \end{pmatrix}(t) = {} & \Phi(t, 0)\, \Delta^{-1} \left\{ v - \sum_{j=1}^{N} B_j \sum_{l=0}^{j-1} \Phi(j, l+1)\, h(l) \right\} \\ & + \sum_{j=0}^{t-1} \Phi(t, j+1)\, h(j) \qquad \text{for } t > 0 \end{aligned}$$

and

$$H\begin{pmatrix} h \\ v \end{pmatrix}(0) = \Delta^{-1} \left\{ v - \sum_{j=1}^{N} B_j \sum_{l=0}^{j-1} \Phi(j, l+1)\, h(l) \right\}.$$

The map H defined above is clearly linear and continuous. We will denote its norm by $\|H\| = \sup \{ \|H\binom{h}{v}\| \mid \|\binom{h}{v}\| = 1 \}$.

3. NONLINEAR SYSTEMS

We shall now see that the properties of existence and uniqueness of solutions for the linear boundary value problem (1.5)–(1.6) are preserved under "small" nonlinear perturbations of both the difference equation and the boundary conditions.

THEOREM 3.1. *Suppose the matrix Δ is nonsingular and that there exist constants $K, K_0,\ldots, K_N$ such that:*

(i) $|f(t, x)-f(t, y)| \leqslant K\,|x-y|$ *for all* x, y *in* $\mathbb{R}^n$ *and*

(ii) $|g(x_0,\ldots, x_N)-g(y_0,\ldots, y_N)| \leqslant \sum_{j=0}^{N} K_j\,|x_j-y_j|$ *for* $x_j, y_j \in \mathbb{R}^n$, $j=0,\ldots, N$.

If $|\lambda| < [\|H\|(K+\sum_{j=0}^{N} K_j)]^{-1}$ there exists a unique solution to

$$x(t+1)=A(t)\,x(t)+h(t)+\lambda f(t, x(t)) \tag{3.1}$$

subject to

$$\sum_{j=0}^{N} B_j x(j)=v+\lambda g(x(0)),\ldots, x(N)). \tag{3.2}$$

Furthermore, the solution can be obtained by iterations.

Proof. We consider the mapping $N\colon X \to Z \times \mathbb{R}^n$ given by

$$N(\phi)=\begin{pmatrix} N_1(\phi)\\ N_2(\phi)\end{pmatrix} \in Z \times \mathbb{R}^n$$

where

$$(N_1\phi)(t)=f(t, \phi(t)), \qquad t=0, 1,\ldots, N$$

and

$$N_2(\phi)=g(\phi(0), \phi(1),\ldots, \phi(N)) \in \mathbb{R}^n.$$

We see that if ϕ and ψ belong to X,

$$\begin{aligned}
&\|N(\phi)-N(\psi)\|\\
&\quad= \max_{t\in\{0,1,\ldots,N\}} |N_1(\phi)(t)-N_2(\psi)(t)|+|N_2(\phi)-N_2(\psi)|\\
&\quad= \max_{t\in\{0,1,\ldots,N\}} |f(t, \phi(t))-f(t, \psi(t))|\\
&\qquad+|g(\phi(0),\ldots,\phi(N))-g(\psi(0),\ldots, \psi(N))|\\
&\quad\leqslant \max_{t\in\{0,1,\ldots,N\}} K|\phi(t)-\psi(t)|+\sum_{j=0}^{N} K_j|\phi(j)-\psi(j)|\\
&\quad\leqslant K\|\phi-\psi\|+\|\phi-\psi\|\sum_{j=0}^{N} K_j=\left(K+\sum_{j=0}^{N} K_j\right)\|\phi-\psi\|.
\end{aligned}$$

It is clear that solving (3.1)–(3.2) is equivalent to solving

$$x = \lambda H(N(x)) + H\begin{pmatrix} h \\ v \end{pmatrix}.$$

We see that for ϕ and ψ in X

$$\|\lambda H(N(\phi)) - \lambda H(N(\psi))\| \leqslant |\lambda| \, \|H\| \left(K + \sum_{j=0}^{N} K_j \right) \|\phi - \psi\|.$$

Hence, for $|\lambda| < [\|H\|(K + \sum_{j=0}^{N} K_j)]^{-1}$ we see that the map $x \mapsto \lambda H(N(x)) + H(\begin{smallmatrix} h \\ v \end{smallmatrix})$ is a contraction. Since X is a Banach space the contraction mapping principle [12] establishes the existence of a unique $\bar{x}$ in X such that

$$\bar{x} = \lambda H(N(\bar{x})) + H(\begin{smallmatrix} h \\ v \end{smallmatrix}).$$

This fixed point is the unique solution of the boundary value problem (3.1)–(3.2). It is a direct consequence of the contraction mapping principle that if x^1 is an arbitrary element of X and x^m is defined recursively by

$$x^{m+1} = \lambda H(N(x^m)) + H(\begin{smallmatrix} h \\ v \end{smallmatrix})$$

the sequence $\{x^m\}$ converges to $\bar{x}$.

We now consider the boundary value problem

$$x(t+1) = f(t, x(t)) \tag{3.3}$$

subject to

$$g(x(0), ..., x(N)) = 0, \tag{3.4}$$

where $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $g\colon \mathbb{R}^{n(N+1)} \to \mathbb{R}^n$.

DEFINITION. We say that a solution $\bar{x}(\cdot)$ of (3.3)–(3.4) is isolated if there exists a $\delta > 0$ such that if $x(\cdot)$ is another solution of (3.3)–(3.4), $\max_{t \in \{0,1,...,N+1\}} |x(t) - \bar{x}(t)| \geqslant \delta$.

THEOREM 3.2. *Suppose f and g are continuously differentiable, that $\bar{x}(\cdot)$ is a solution of* (3.3)–(3.4), *and let $\Psi(t, l)$ be the state-transition matrix of the system*

$$y(t+1) = \frac{\partial f}{\partial x}(t), \bar{x}(t)) \, y(t).$$

If the matrix D defined by

$$D = \sum_{k=0}^{N} \frac{\partial g}{\partial x_k} (\bar{x}(0),..., \bar{x}(N)) \, \Psi(k, 0)$$

is nonsingular, then $\bar{x}(\cdot)$ *is an isolated solution of* (3.3)–(3.4).

Note. $\partial f/\partial x$ represents the Jacobian matrix of f with respect to x and $\partial g/\partial x_k$ denotes the Jacobian of the mapping $x_k \mapsto g(x_0,..., x_k,..., x_N)$, where $x_j \in \mathbb{R}^n$ for each j and x_j is fixed for $j \neq k$.

Proof. Since f is continuously differentiable it follows that if ϕ is an element of X and $\varepsilon > 0$ is given there exist constants $\delta_0,..., \delta_N$ such that

$$|f(j, \phi(j) + h(j)) - f(j, \phi(j)) - \frac{\partial f}{\partial x} (j, \phi(j)) \, h(j)| < \varepsilon \, |h(j)|$$

for $j = 0, 1,..., N$, provided $|h(j)| < \delta_j$. From this it follows that if h is an element of X and $\|h\| < \delta = \min_{j=0,...,N} \delta_j$,

$$\max_{t \in \{0,...,N\}} |f(t, \phi(t) + h(t)) - f(t, \phi(t)) - \frac{\partial f}{\partial x} (t, \phi(t)) \, h(t)| < \varepsilon \|h\|.$$

The above estimates show that if $\mathscr{F}_1 \colon X \to Z$ is given by

$$\mathscr{F}_1(\phi)(t) = \phi(t+1) - f(t, \phi(t))$$

the Fréchet derivative of $\mathscr{F}_1$ is given by

$$\mathscr{F}'_1(\phi)(h)(t) = h(t+1) - \frac{\partial f}{\partial x} (t, \phi(t)) \, h(t)$$

for $h \in X$ and $t = 0, 1,..., N$. Similarly, if $\mathscr{F}_2 \colon X \to \mathbb{R}^n$ is defined by

$$\mathscr{F}_2(\phi) = g(\phi(0),..., \phi(N))$$

we see that for h in X

$$\mathscr{F}'_2(\phi)(h) = \sum_{j=0}^{N} \frac{\partial g}{\partial x_j} (\phi(0),..., \phi(N)) \, h(j).$$

Hence, if $\mathscr{F} \colon X \to Z \times \mathbb{R}^n$ is given by

$$\mathscr{F}(\phi) = \begin{pmatrix} \mathscr{F}_1(\phi) \\ \mathscr{F}_2(\phi) \end{pmatrix}$$

then $\mathscr{F}$ is continuously Fréchet differentiable and

$$\mathscr{F}'(\phi)(h) = \begin{pmatrix} \mathscr{F}'_1(\phi)(h) \\ \mathscr{F}'_2(\phi)(h) \end{pmatrix}$$

for ϕ and h in X.

Therefore if $\bar{x}(\cdot)$ is a solution of (3.3)–(3.4), $h(\cdot)$ is an element of X, and $\binom{y(\cdot)}{v}$ belongs to $Z \times \mathbb{R}^n$, we have that

$$\mathscr{F}'(\bar{x})(h) = \begin{pmatrix} y(\cdot) \\ v \end{pmatrix}$$

if and only if

$$h(t+1) = \frac{\partial f}{\partial x}(t, \bar{x}(t))\, h(t) + y(t), \qquad t = 0, 1, \ldots, N, \tag{3.5}$$

and

$$\sum_{j=0}^{N} \frac{\partial g}{\partial x_j}(\bar{x}(0), \ldots, \bar{x}(N))\, h(j) = v. \tag{3.6}$$

Since the matrix D is nonsingular, a direct application of Lemma 2.1 shows that for each $\binom{y(\cdot)}{v}$ in $Z \times \mathbb{R}^n$ there exists a unique solution of (3.5)–(3.6). Hence $\mathscr{F}'(\bar{x})$ maps one-to-one and onto $Z \times \mathbb{R}^n$ and by the inverse function theorem [12, 14] it follows that $\bar{x}(\cdot)$ is an isolated solution of (3.3)–(3.4).

This theorem extends Theorem 5.1 in Agarwal [2] to the case of nonlinear boundary conditions.

Definition. An element $\bar{x}(\cdot)$ of X is called an ε-approximate solution of (3.3)–(3.4) if

$$|g(\bar{x}(0), \ldots, \bar{x}(N))| + \max_{t \in \{0, \ldots, N\}} |\bar{x}(t+1) - f(t, \bar{x}(t))| < \varepsilon.$$

Definition. An ε-approximate solution $\bar{x}(\cdot)$ of (3.3)–(3.4) is said to be regular if the matrix $D = \sum_{j=0}^{N} (\partial g/\partial x_j)\ (\bar{x}(0), \ldots, \bar{x}(N))\Psi(j, 0)$ is nonsingular, where $\Psi(k, l)$ is the state-transition matrix for $y(t+1) = (\partial f/\partial x)(t, \bar{x}(t))y(t)$.

From Lemma 2.1 we see that if $\bar{x}(\cdot)$ is a regular ε-approximate solution of (3.3)–(3.4) then for any $\binom{h(\cdot)}{v}$ in $Z \times \mathbb{R}^n$ there exists a unique $x = H(h(\cdot))$ in X that solves

$$x(t+1) = \frac{\partial f}{\partial x}(t, \bar{x}(t))\ x(t) + h(t)$$

subject to

$$\sum_{j=0}^{N} \frac{\partial g}{\partial x_j}(\bar{x}(0),..., \bar{x}(N))\, h(j) = v$$

and H is given by

$$H\begin{pmatrix} h(\cdot) \\ v \end{pmatrix}(t) = \Phi(t, 0)\left\{\sum_{j=0}^{N} \frac{\partial g}{\partial x_j}(\bar{x}(0),..., \bar{x}(N)\, \Phi(j, 0)\right\}^{-1}$$
$$\times\left\{v - \sum_{j=1}^{N} \frac{\partial g}{\partial x_j}(\bar{x}(0),..., \bar{x}(N)) \sum_{l=0}^{j-1} \Phi(j, l+1)\, h(l)\right\}$$
$$+ \sum_{j=0}^{t-1} \Phi(t, j+1)\, h(j) \qquad \text{for } t > 0;$$

and

$$H\begin{pmatrix} h(\cdot) \\ v \end{pmatrix}(0) = \left\{\sum_{j=0}^{N} \frac{\partial g}{\partial x_j}(\bar{x}(0),..., \bar{x}(N))\, \Phi(j, 0)\right\}^{-1}$$
$$\times\left\{v - \sum_{j=1}^{N} \frac{\partial g}{\partial x_j}(\bar{x}(0),..., \bar{x}(N)) \sum_{l=0}^{j-1} \Phi(j, l+1)\, h(l)\right\}.$$

THEOREM 3.3. *Let* $\bar{x}(\cdot)$ *be a regular,* ε*-approximate solution of* (3.3)–(3.4) *and suppose there exist numbers* $\delta > 0$ *and* $0 < k < 1$ *such that*

(i) $\|H\|\varepsilon \leqslant \delta(1-k)$ and

(ii) $$\max_{t \in \{0,1,...,N\}} \left\{\left\|\left(\frac{\partial f}{\partial x}(t, \bar{x}(t)) - \frac{\partial f}{\partial x}(t, \phi(t))\right)\right\|\right.$$
$$\left. + \left\|\sum_{j=0}^{N}\left(\frac{\partial g}{\partial x_j}(\bar{x}(0),..., \bar{x}(N)) - \frac{\partial g}{\partial x_j}(\phi(0),..., \phi(N))\right)\right\|\right\} \leqslant k\|H\|^{-1}$$

for all ϕ *in* X *such that* $\|\phi - \bar{x}\| \leqslant \delta$. *Then there exists one and only one solution* x *of* (3.3)–(3.4) *that satisfies* $\|x - \bar{x}\| \leqslant \delta$. *Moreover, this solution can be obtained by iterations.*

Note. The symbol $\|\cdot\|$ denotes the operator norm for bath, matrices, and operators.

Proof. We use $\Omega_\delta = \{\phi \in X \mid \|\phi - \bar{x}\| \leqslant \delta\}$ and we define $F\colon X \to Z \times \mathbb{R}^n$ by

$$F(\phi) = \begin{pmatrix} F_1(\phi) \\ F_2(\phi) \end{pmatrix}$$

where $F_1\colon X \to Z$ is given by $F_1(\phi)(t) = \phi(t+1) - f(t, \phi(t))$ and $F_2\colon X \to \mathbb{R}^n$ is given by $F_2(\phi) = g(\phi(0),..., \phi(N))$.

From the above assumptions it is clear that the Fréchet derivative of F at $\bar{x}$ is invertible and that

$$DF(\bar{x})^{-1}\begin{pmatrix} h \\ v \end{pmatrix}(t) = H\begin{pmatrix} h \\ v \end{pmatrix}(t).$$

For each ϕ in Ω_δ we define $T: \Omega_\delta \to X$ by $T(\phi) = \phi - HF(\phi)$. The Fréchet derivative of T at ϕ is given by

$$DT(\phi) = I - HDF(\phi) = HDF(\bar{x}) - HDF(\phi).$$

Therefore, if $u \in X$ and $\|u\| = 1$ $\|DT(\phi)(u)\| \leqslant \|H\|$ $\|DF(\bar{x})(u) - DF(\phi)(u)\| \leqslant \|H\| \max_{t \in \{0,..., N\}} \{\|((\partial f/\partial x)(t, \bar{x}(t)) - (\partial f/\partial x)(t, \phi(t)))\| + \|\sum_{j=0}^{N} ((\partial g/\partial x_j)(\bar{x}(0),..., \bar{x}(N)) - (\partial g/\partial x_j)(\phi(0),..., \phi(N)))\|\} \leqslant \|H\|$ $k\|H\|^{-1} = k$.

Using the mean value theorem, we have that for ϕ_1, ϕ_2 in $\Omega_\delta \|T(\phi_1) - T(\phi_2)\| \leqslant k\|\phi_1 - \phi_2\|$. Also, for $\phi \in \Omega_\delta$,

$$\begin{aligned}\|T(\phi) - \bar{x}\| &\leqslant \|T(\phi) - T(\bar{x})\| + \|T(\bar{x}) - \bar{x}| \\ &\leqslant k\|\phi - \bar{x}\| + \|HF(\bar{x})\| \leqslant k\|\phi - \bar{x}\| + \|H\|\varepsilon \leqslant \delta.\end{aligned}$$

Hence, T maps Ω_δ into Ω_δ and it is a contraction. Therefore there exists a unique fixed point of T in Ω_δ and hence a unique solution of (3.3)–(3.4) in Ω_δ. It is clear that this solution, x, can be obtained as $x = \lim_{j \to \infty} x_j$, where $x_0 = \bar{x}$ and $x_{j+1} = T(x_j)$.

This result extends Theorems 5.1 and 5.3 of Agorwal [2] to the case of nonlinear boundary conditions.

4. Examples

The following example illustrates some of the qualiative differences that exist between the solutions of boundary value problems in differential equations and those of its discretized version. We consider

$$\dot{x} = f(x) \tag{4.1}$$

subject to

$$Bx(0) + Dx(1) = 0 \tag{4.2}$$

where f is continuously differentiable and B and D are constant n by n matrices such that $B + D$ is nonsingular.

It is well known [6, 7, 11, 18] that under these mild conditions the boundary value problem (4.1)–(4.2) may have no solution, a unique solution, or multiple solutions.

A discretized version of this problem with equal step lengths $\lambda > 0$ is given by

$$y(t+1) = y(t) + \lambda f(y(t)) \tag{4.3}$$

subject to

$$By(0) + Dy(N) = 0 \tag{4.4}$$

where $N = \lambda^{-1}$.

It is clear that the state transition matrix for (4.3)–(4.4) is given by $\Phi(k, l) = I$ (the identity matrix). Hence, we see that $B + D\Phi(N, 0)$ is nonsingular and therefore there exists a $\lambda_0 > 0$ such that for all $|\lambda| \leqslant \lambda_0$ there is a unique solution of (4.3)–(4.4).

We now consider a two-dimensional "weakly" nonlinear system of the form

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \lambda \begin{pmatrix} f_1(x_1(t), x_2(t)) \\ f_2(x_1(t), x_2(t)) \end{pmatrix}$$

subject to

$$x_1(0) = v_1 + \lambda g_1(x_1(0), x_2(0), \ldots, x_1(N), x_2(N))$$

$$x_2(1) + x_1(N) = v_2 + \lambda g_2(x_1(0), x_2(0), \ldots, x_1(N), x_2(N))$$

where we assume that λ, v_1, and v_2 are real numbers and the functions $f = \binom{f_1}{f_2}$ and $g = \binom{g_1}{g_2}$ satisfy the smoothness hypothesis of Theorem 3.1.

We see that this problem is of the form

$$x(t+1) = Ax(t) + \lambda f(x(t)) \tag{4.5}$$

subject to

$$\sum_{j=0}^{N} B_j x(j) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \lambda g(x(0), \ldots, x(N)) \tag{4.6}$$

where

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and $B_j = 0_{n \times n}$ for $j \neq 0, 1, N$. It is straightforward to verify that $\Phi(t, 0) = \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}$ and hence that

$$\sum_{j=0}^{N} B_j \Phi(j, 0) = \begin{pmatrix} 1 & 0 \\ -N & N+2 \end{pmatrix}.$$

Since this matrix is nonsingular it follows from Theorem 3.1 that there exists $\lambda_0 > 0$ such that for each λ such that $|\lambda| \leqslant \lambda_0$ there exists a unique solution of (4.5)–(4.6).

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